

A ROBUST ALGORITHM FOR EARTHWORKS VOLUME COMPUTATIONS

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ABSTRACT

The quest for the most mathematically accurate algorithm to resolve volumes between complex planimetric and vertically overlapping surfaces is discussed. A complete algorithm for surface-to-surface digital terrain modeling, called the ‘composite method,’ is given. The algorithm is applied to two real-world earthworks volumetric problems: (1) performing simultaneous cut and fill operations and (2) evaluating the volumes of complex material stockpiles. The author makes the argument that this is the only accurate method for determining the volumes between two multi-planar interacting surfaces.

1.0 INTRODUCTION

In 2009 the author, a Land Surveyor with 20 years’ experience (at the time), was engaged by two clients separately with two apparently different problems that turned out to have one solution, though this was not immediately obvious. Rather than resort to plugging the data into commercial applications for ready results, the author decided to investigate the algorithms used by these commercial packages and to take look inside their ‘black boxed’ routines that most surveyors blindly put their faith in without question. This proved to be a fruitful exercise as the author was able to detect procedural blunders made by the first client’s contractor and by the second client’s surveyors that resulted in erroneous volume estimates being given to both clients. These procedural errors were made from a lack of understanding of the Euclidean geometry involved in digital terrain modeling that the author hopes to explore and explain in the following sections.

2.0 A TALE OF TWO CLIENTS

2.1 Problem 1: Resolving a Disputed Earthworks Claim

Client A requested an independent verification of the cut and fill volumes from earthworks carried out by Contractor X on Client A’s behalf on a site in the south-western part of Trinidad of the twin-island State of The Republic of Trinidad & Tobago. The objective was to prepare an undulating site for development by simultaneously cutting and filling the areas above and below, respectively, a pre-determined final surface. Contractor X provided the data from two topographical surveys which were carried out before and after the earthworks respectively. Contractor X stated that (a) the two surveys were executed using the same planimetric grid and height datums and (b) no material was brought onto nor removed from the site. The separate cut and fill volumes were required for correct payment to Contractor X. There were specific rates for cutting and filling operations. Client A was concerned that Contractor X’s final volumes far exceeded the values estimated by the project engineers based on the publicly available Lands & Surveys digital mapping and the proposed design elevation.

2.2 Problem 2: Resolving the Volume of Stockpiles in Complex Sheds

Client B required the volumes of stockpiled aggregate material stored in sheds. The catch was that the bases of the sheds were not simple flat floors but were in-fact oddly-shaped and in some cases, non-symmetric. To add insult to injury, it was not possible to survey the actual bases of the sheds as the thickness of the material *in situ* made those areas inaccessible. Client B instead grudgingly provided the architectural drawings for the shed bases.

3.0 BUILDING A SOLUTION ONE TRIANGLE AT A TIME

3.1 Starting with the Basics of Volume Computation

Before jumping directly into methods for resolving complex volume problems, the author decided to start with the basics of simple volume computations and build up from there. The first task was modeling a surface from 3D point data: how to go from points to lines and then 3-dimensional triangles to make a Triangular Irregular Network (TIN).

3.2 Generating TINs by Delaunay Triangulation

A pre-requisite to obtaining a Digital Terrain Model (DTM) [1] comprising triangulated points from a given set of XYZ data is a method for connecting the points by lines to form unique non-overlapping triangles with the preference that they be as close to equilateral as possible. Such a method was developed by Boris Delaunay, a Russian mathematician, in 1934. Ironically the method predates the earliest electronic computers by a decade [2] but due to potentially large data sets comprising thousands of points, it can only be practically implemented programmatically.

The single condition for Delaunay Triangulation is that there must be no other points planimetrically inside of the circumcircle of any given triangle. An unwanted side-effect of the algorithm is the generation of invalid “flat” triangles between un-correlated boundary points. A “flat” triangle is defined by having one internal angle greater than 160° (Fig.1). For our purposes the method is two-dimensional and the triangulation is not affected by a point’s elevation. There is also an advanced triangulation method utilizing circumspheres but it is not required here.

A C language subroutine for Delaunay Triangulation was written in 1989 by Paul Bourke, a Research Associate Professor from the University of Western Australia, who put the code into the public domain [3]. Professor Bourke’s subroutine is fast and robust and is used extensively by the author who has also re-written it in AutoLISP to run within AutoCAD.

4.0 SIMPLE CUT OR FILL VOLUMES FROM A DATUM PLANE

The volume between a surface represented by a TIN and a fixed datum plane which is arbitrarily lower than the smallest elevation in the model or arbitrarily higher than the highest elevation, representing a cut and a fill respectively (Fig.2), can be easily

computed by summing the volume between each triangular plane and the datum plane (Fig.3). The value dH_1 is the difference between the elevations of the 3D triangle point and the datum plane, dH_2 and dH_3 are similarly defined. The area is the planimetric area of the 3D triangle as projected vertically onto the datum plane. The volume of the composite solid is therefore the mean dH multiplied by the planimetric area.

5.0 CUT OR FILL BETWEEN NON-INTERSECTING SURFACES: THE REFERENCE DATUM METHOD

A two-step computation is required when there are two irregular non-intersecting surfaces (Fig.4). In this example, a cut volume is computed by determining the volume V_1 between Surface1 and the datum plane and the volume V_2 between Surface2 and the datum plane. The required cut volume is obtained by from $V_1 - V_2$. This is the 'reference datum' method.

6.0 CUT AND FILL VOLUMES BETWEEN TWO VERTICALLY INTERSECTING SURFACES

6.1 A Different Kind of Problem: Space Invaders

When there are two vertically intersecting surfaces (Fig.5) the 'reference datum' method discussed in Section 5.0 will fail because that method yields a single cut or fill value. Here we require both a cut and a fill volume simultaneously. Another algorithm is required that will determine separately and accurately exactly by how much Surface1 had to be cut to become Surface2 where Surface1 was higher in elevation and by how much Surface1 had to be filled to become Surface2 where Surface1 was lower in elevation. This is a different kind of problem and it is the crux of the assignment given in Section 2.1.

6.2 Direct Surface-to-Surface Comparison: The Composite Method

A search in the standard surveying literature for a solution to the problem in Section 6.1 proved futile. An online search was more successful but not from academic sources as expected. A few websites for commercial earthworks software had succinct descriptions of the 'composite method.' This method was described as follows: find all the planimetric triangles formed by overlaying the two surfaces and compute a third surface, the isopach surface, which is a purely mathematically TIN where the z-values of the triangle corners are the height differences between the two original surfaces. The cut and fill volumes are computed by summing those parts of the isopach surface which are above and below the zero datum plane respectively.

6.3 Programming the Composite Method

The terse description for the composite method given in Section 6.2 proved to be deceptively complex to implement programmatically. After many trials and errors the author was able to write a program in the C language that eventually yielded results for cut and fill volumes which were identical to the results from commercial packages. The

key steps for programming the algorithm are given in Fig.6 in pseudocode. Besides writing code for the basic computational geometry, additional routines were introduced to increase the program's efficiency since there is the potential for having to detect and process thousands of overlapping triangle intersections depending on the size of the input TINs.

6.4 Overlapping Triangles

In overlaying Surface1 and Surface2, a myriad of overlapping sub-areas are formed which can take on shapes of 3 to 6 sides forming triangles, quadrilaterals, pentagons and hexagons which can be further sub-divided into 1 to 4 sub-triangles (Fig.7). Each sub-triangle is an isopach triangle.

6.5 Computing the Isopach Triangles' Z-Values

The two sub-triangles from Surface1 and Surface2 respectively are identical planimetrically with the same (x, y) values at their corresponding corners and both have the same planimetric area. They differ only in their z-values. In order to create a single isopach triangle from two overlapping triangles in two different planes, we have to use subtraction (Fig.8).

The z-values of the corner points of the isopach triangle are computed by subtracting the z-values in Surface2 from the corresponding points' z-values in Surface1 to get $dH1$, $dH2$ and $dH3$ which may be independently positive or negative. The three values $dH1$, $dH2$ and $dH3$ become the z-values of the isopach triangle's corners relative to the zero plane. The 3D co-ordinates for the isopach triangle would then be $(x1, y1, dH1)$, $(x2, y2, dH2)$ and $(x3, y3, dH3)$.

6.6 Computing Cut and Fill Volumes from the Isopach Surface

When all three z-values of the isopach triangle's corners are positive we get a positive volume which is a cut volume and the absolute value is added to the cut summation. When all three differences are negative we get a negative volume which is a fill volume and the absolute value is added to the fill summation. The third remaining option is the isopach triangle is intersected by the zero plane (Fig.9).

In most cases the intersection of an isopach triangle with the zero plane results in the formation of a quadrilateral and a triangle. The solid formed by either shape can be above or below the datum plane which gives the cut and fill volumes respectively. The absolute value of each volume is added to the cut and fill summations respectively.

7.0 THE MERCHANT OF EARTHWORKS: IF WE CUT DO WE NOT FILL?

7.1 The Numbers Racket: Skimming off the Top

Fig.10 shows the pre- and post-topographic surveys for Client A's site superimposed and in side-view while Fig.11 shows the program's output. These values were found to be

approximately 30% less than Contractor X's claims. Additionally the cut and fill volumes did not equate which should have happened since excavated material never left the site as stated in Section 2.1. Rather than hastily jumping to a conclusion of subterfuge on Contractor X's part, some of the possible explanations for the differences and imbalances were examined and subsequently included in the full report to Client A.

7.2 It was the Worst of TINs: Digital Terrain Modeling in the Age of Foolishness

Taking a close look at Contractor X's volume computations revealed three unintentional blunders which resulted in the size of the results being inflated. The first was the inadvertent inclusion of control points that lay outside of the site boundary (Fig.12). The additional 3D triangles thus generated would add a considerable invalid volume. Similarly the second error was the failure to exclude the "flat" triangles discussed in Section 3.2. Thirdly the total area covered by the two surveys was used by Contractor X's non-discerning software. Only areas of planimetric overlap between the two TINs are valid but by including non-overlapping area, spurious values were added to Contractor X's final figures.

7.3 Low Blow: A Geo-Technical Knockout

The imbalance between the cut and fill values was explained by the nature of the site terrain: the very low-lying areas were swampy marshland that required more material to fill than the firmer higher ground that lay below the design datum. Geotechnical testing resulted in the repeated filling and compaction of these areas. In addition the pure volume computation does not take material bulking when excavated into consideration as it can only process the point data provided in terms of pure Euclidean geometry.

7.4 Settlement: An ISO 2009 Pact

The final outcome between Client A and Contractor X was a negotiated settlement for a volume greater than the author's computation but less than Contractor X's original claim, somewhat akin to an isopach mean-value analogy.

8.0 PLAYING THE STOCKPILE MARKET: BROUGHT LOW – TOLD HIGH

8.1 When Bad Stockpiles Happen to Good Methods

Simple material stockpiles can be represented by two TINs, one for the pile mound (Surface1) and another for the base (Surface2). By utilizing the 'reference datum' method discussed in Section 5.0 and shown in Fig.4, we obtain the stockpile volume using the difference of two volumes from a reference plane. For stockpiles stored in sheds with complex (non-flat) floors (Fig.13), the method from Section 5.0 would fail because the material shown in the grey area would be excluded from the computation since the automatically generated base TIN (Surface2) would be 'unaware' of the true shape of the pile sub-surface.

8.2 Thinking Outside the TIN Box: Non-Delaunay Triangulation

The ‘composite method’ is well-suited to vertically intersecting surfaces but it then occurred to the author that the method yields mathematically accurate results when applied to any two overlapping surfaces that are triangulated, Delaunay or otherwise. The shed floors can be modeled not by automatic Delaunay triangulation but by manually creating 3D triangular planes (called 3DFaces in AutoCAD) based on the dimensions from the architectural drawings provided in Section 2.2. Once the shed floor DTMs were created they could then be re-used for future stockpile surveys in those sheds.

The solution is to make the measured pile Surface1 (*sans* automatic pile base TIN) and the DTM of the corresponding shed floor Surface2 (Fig.14). The surface-to-surface ‘composite method’ is then applied: there would be a positive cut value which is the stockpile volume and a zero fill volume since there is no material underneath the shed’s concrete floor. When the program was run and the results delivered to Client B, the feedback was very similar to that from Contractor X: the computed volumes were 25% less than expected. Again possible explanations were sought and reported to Client B.

8.3 It Fell off the Back of a Truck

The single most erroneous concept Client B harboured was equating mass with volume. Whereas mass rigidly obeys the First Law of Thermodynamics (by not disappearing arbitrarily from existence), volume is elusive and is wanted in several states. Materials expand in volume when excavated and transported then revert to a compact state from the combination of heavy moving machinery, gravity and drying when stockpiled. Client B’s material was clay which has a bulking factor of 30% when excavated. This figure closely tallies with the discrepancy between the client’s estimates (which were based on counted truckloads of varying density and capacity) and the computed values. Client B was satisfied with this explanation.

9.0 CONCLUSION

The most mathematically accurate method for computing simultaneous cut and fill volumes between two overlapping surfaces or between a stockpile mound and a complex base is the surface-to-surface ‘composite method.’ Once the algorithm has been programmed successfully, it makes a powerful tool for real-world earthworks volume computations. However this power can go awry if users are not careful with their data and lose sight of how black-boxed algorithms work. Additionally surveyors should appreciate that a pure computation is never a final solution to the problem and that all the extenuating circumstances surrounding an issue should be examined. [■]

REFERENCES

- [1] N. El-Sheimy, 2005. *Digital Terrain Modeling*, Publisher Artech House Remote Sensing Library
- [2] J. Smiley, 2010. *The Man Who Invented the Computer*, Publisher Doubleday
- [3] P. Bourke, 1989. *Efficient Triangulation Algorithm Suitable for Terrain Modelling*. Pan Pacific Computer Conference, Beijing, China

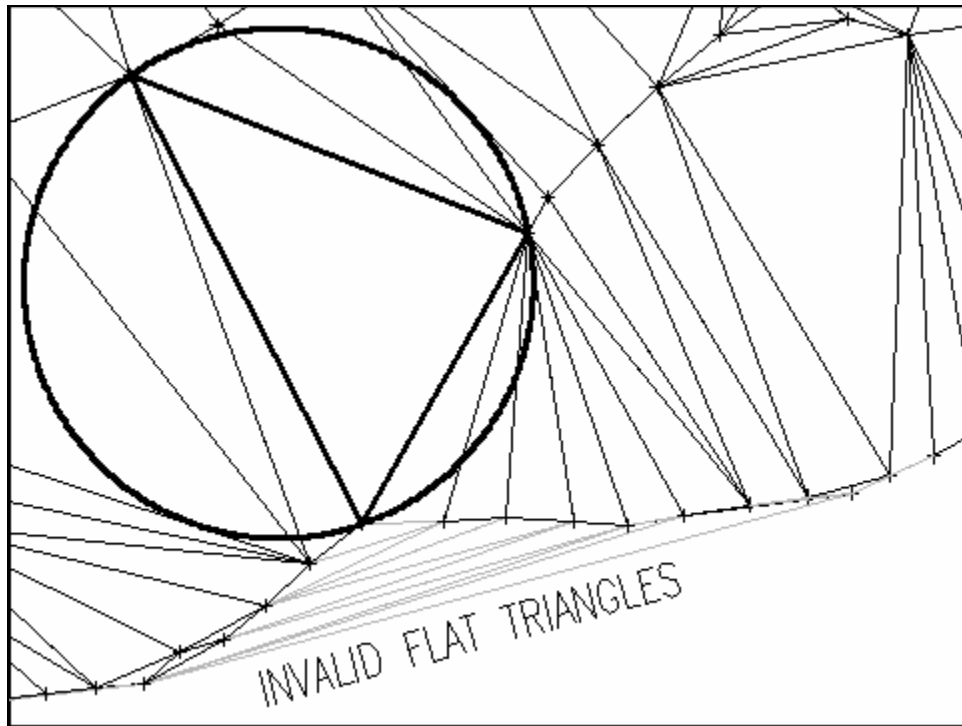


Fig.1 Delaunay Triangulation Showing a Circumcircle and Invalid "Flat" Triangles

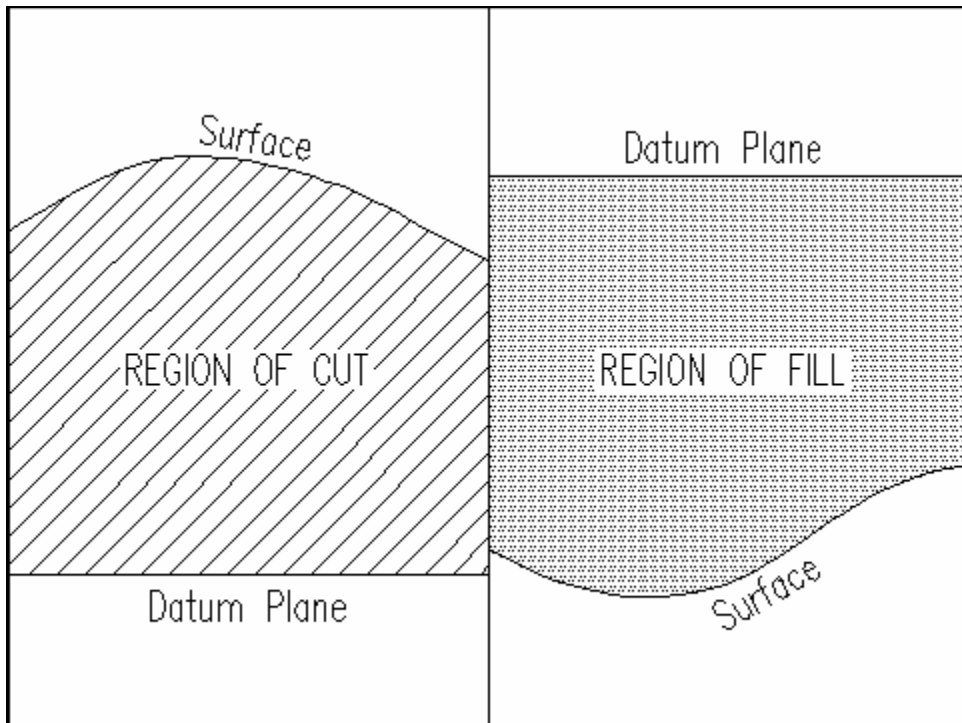


Fig.2 Simple Cut & Fill Volumes Between a Surface and a Reference Datum Plane

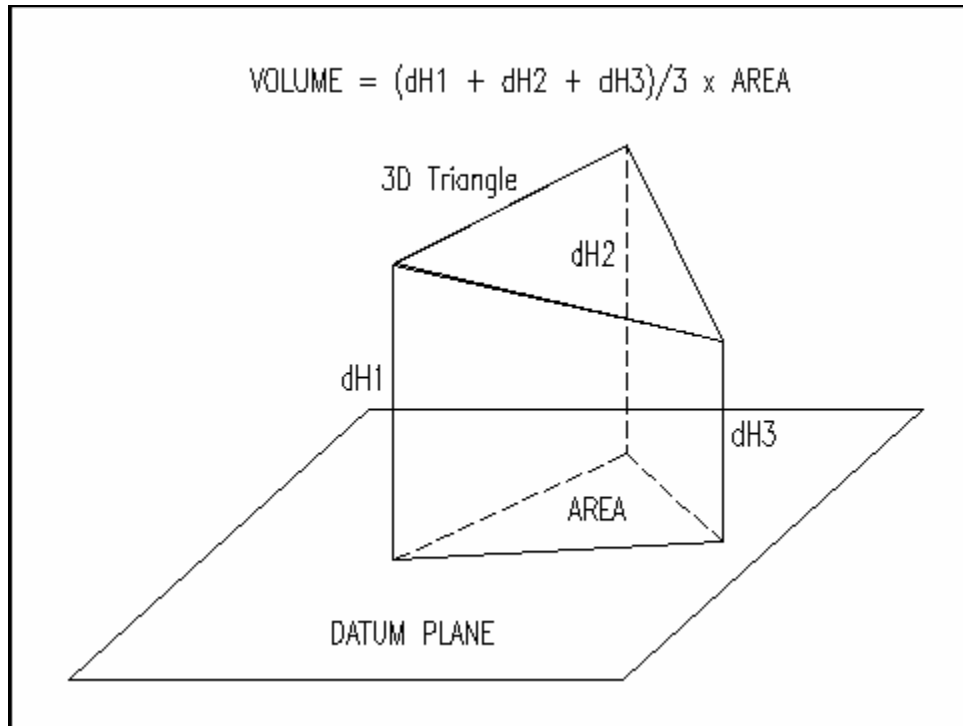


Fig.3 Volume Between a 3D Triangle and a Datum Plane

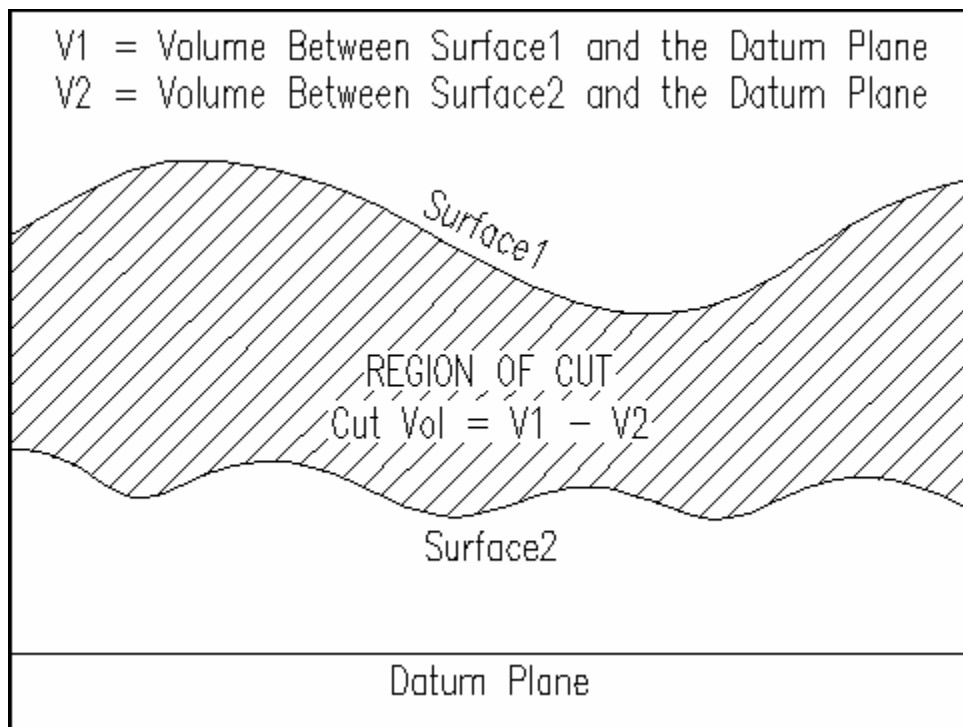


Fig.4 Volume Between Two Non-Intersecting TINs Using a Reference Datum

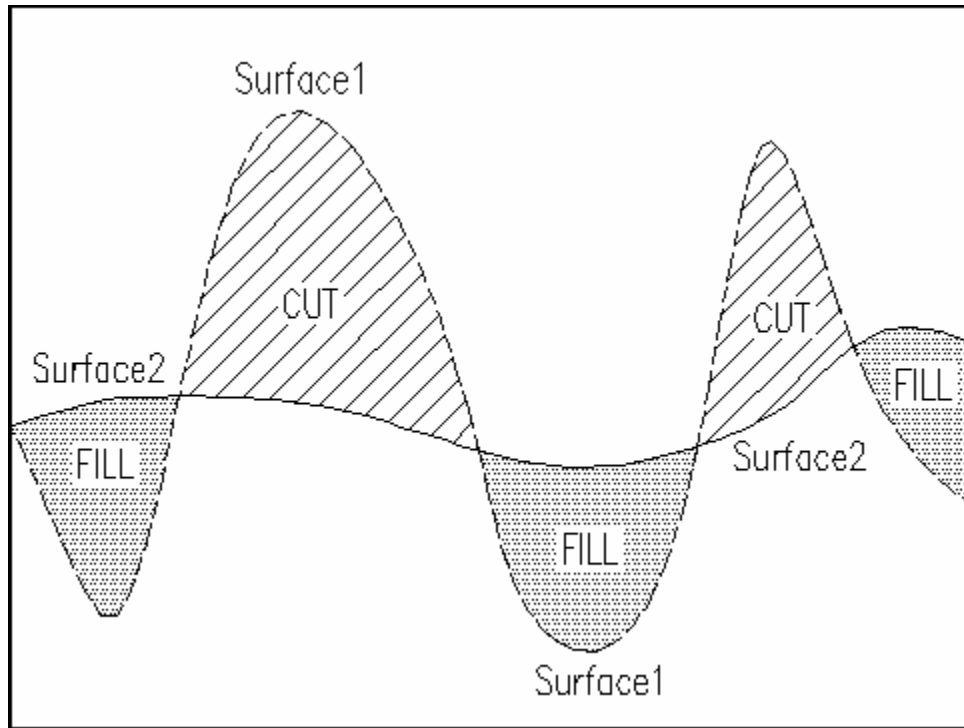


Fig.5 Cut & Fill Volumes Between Two Vertically Intersecting Surfaces

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1  START: Read the 3D (XYZ) Point Data for Surface1
2  Generate TIN1 for Surface1 with Bourke's Delaunay Triangulation Subroutine
3  Eliminate Flat Triangles for Un-Correlated Points at the Outer Boundary
4  Export TIN1 to an External File: P1, P2 & P3 for Each Triangle

5  Read the 3D (XYZ) Point Data for Surface2
6  Generate TIN2 for Surface2 with Bourke's Delaunay Triangulation Subroutine
7  Eliminate Flat Triangles for Un-Correlated Points at the Outer Boundary
8  Export TIN2 to an External File: P1, P2 & P3 for Each Triangle

9  Read in the Triangle Points (P1 P2 P3) from the TIN1 External File
10 Sort Each Triangle's Points so P1, P2 & P3 are in Increasing Eastings (X)
11 Sort All the Triangles so that the P1s are in Increasing Eastings (X)

12 Read in the Triangle Points (P1 P2 P3) from the TIN2 External File
13 Sort Each Triangle's Points so P1, P2 & P3 are in Increasing Eastings (X)
14 Sort All the Triangles so that the P1s are in Increasing Eastings (X)

15 For Each Triangle (T1) in TIN1
16   For Each Triangle (T2) in TIN2
17     IF1 P3(X) in T2 is < P1(X) in T1
18       Yes: T2 is West of T1 therefore No Overlap - Continue
19       No : IF2 P1(X) in T2 is > P3(X) in T1
20         Yes: T2 is East of T1 therefore No Overlap - Stop the Inner Loop
21         No : IF3 Planimetric Overlaps and Intersections between T1 & T2
22           Yes: Compute the Isopach Triangle Elevations = Z1-Z2
23           No : Continue to the Next Triangle in TIN2
24         End IF3
25       End IF2
26     End IF1
27   Next Triangle (T2) in TIN2
28 Next Triangle (T1) in TIN1

29 For Each Triangle (T) in the Isopach Surface
30   IF1 P1(Z) & P2(Z) & P3(Z) > Zero
31     Yes:  $V = (Z1+Z2+Z3)/3$  * Planimetric Area ; Add V to CUT Summation
32     No : IF2 P1(Z) & P2(Z) & P3(Z) < Zero
33       Yes:  $V = (Z1+Z2+Z3)/3$  * Planimetric Area ; Add V to FILL Summation
34       No : Compute Where the Zero Plane Intersects T
35         Add the Volume Vc above the Zero Plane to the CUT Summation
36         Add the Volume Vf below the Zero Plane to the FILL Summation
37     End IF2
38   End IF1
39 Next Triangle (T) in the Isopach Surface

40 Output the CUT VOLUME
41 Output the FILL VOLUME

42 STOP: END

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Fig.6 The 'Composite Method' Algorithm in Pseudocode

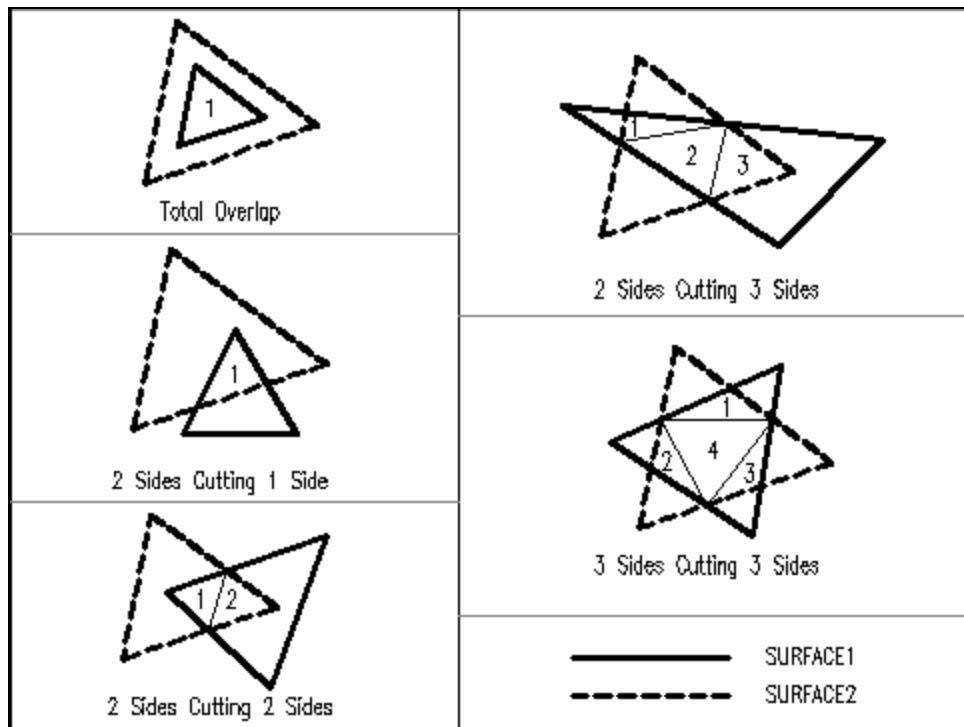


Fig.7 Isopach Triangles Formed from Two Surface Triangles Overlapping

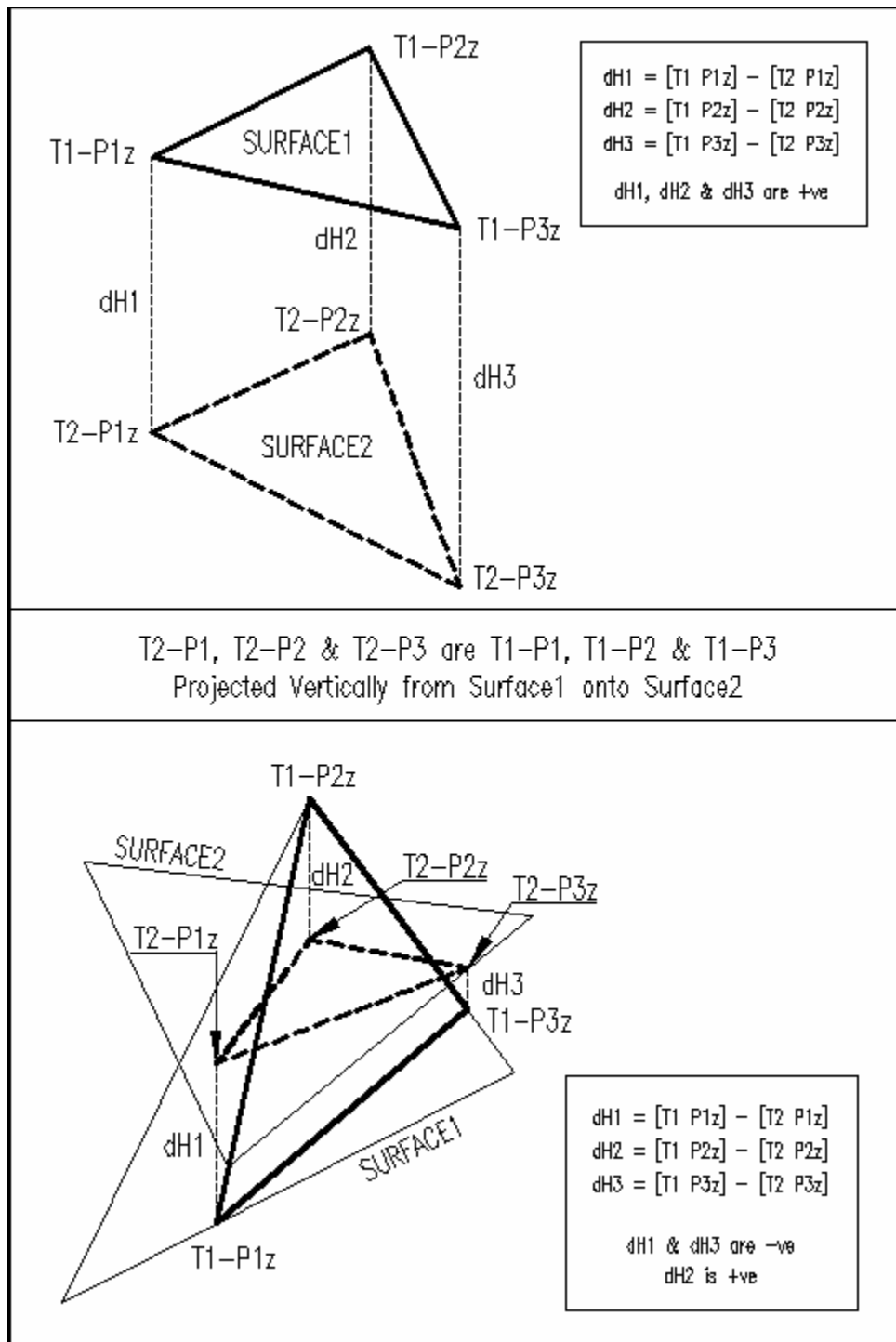


Fig.8 Computing the Z-Values for the Isopach Triangle

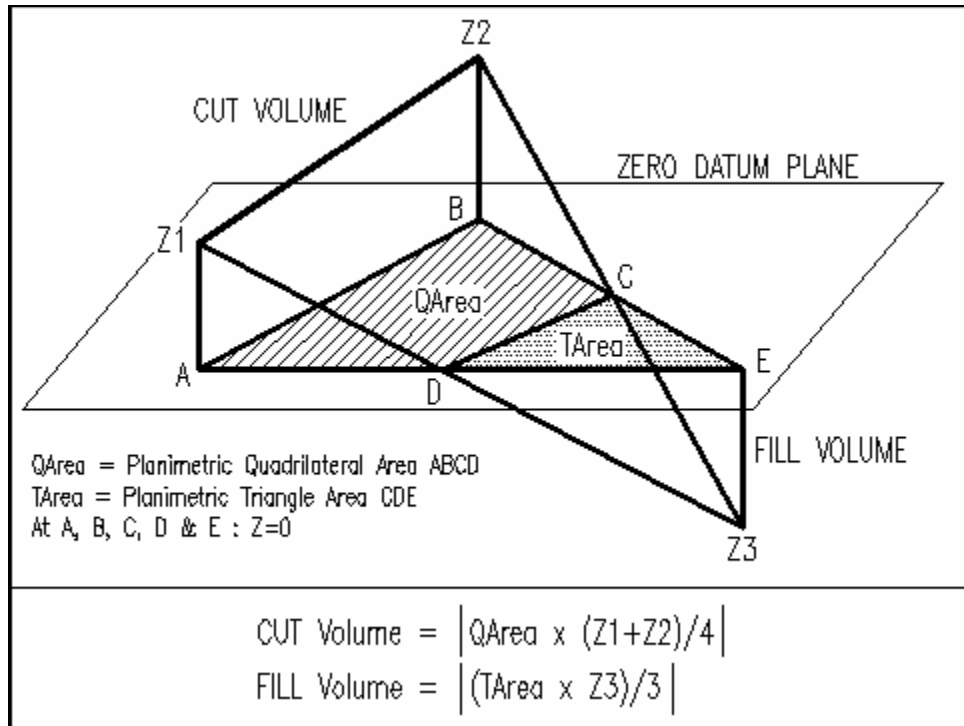


Fig.9 Cut and Fill Volumes from an Isopach Triangle

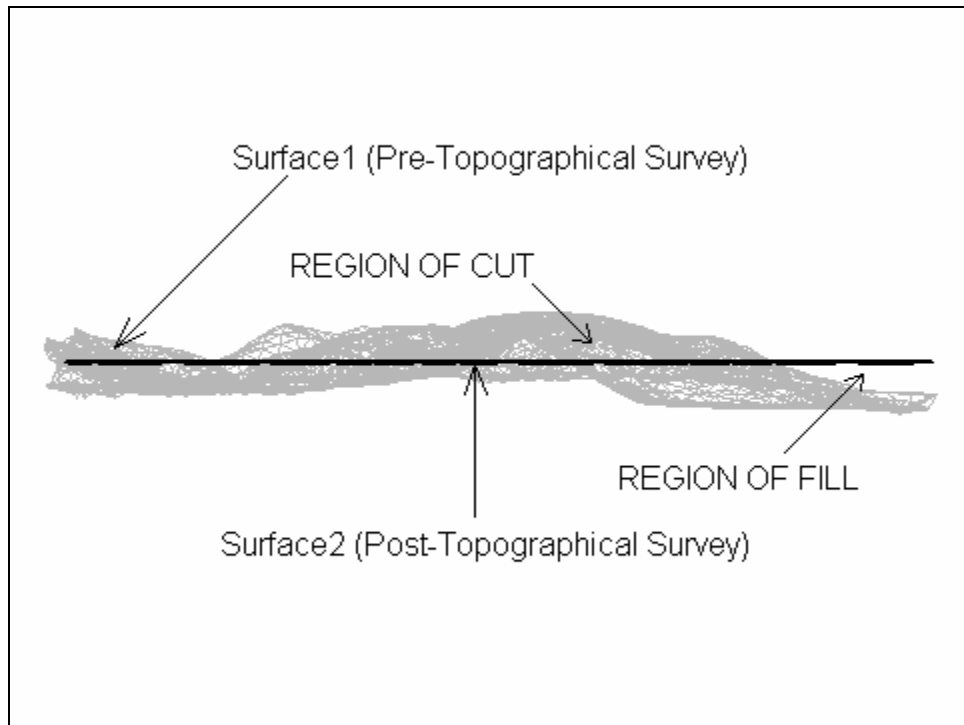


Fig.10 Client A's Two Surfaces Superimposed in Side View

Surface-To-Surface Volume Report

Surface1 Filename: C:\Surface1_tin.txt

Surface2 Filename: C:\Surface2_tin.txt

Surface1 Area = 500538.7 sq. m

Surface2 Area = 478894.5 sq. m

Overlapping Area = 478165.7 sq. m

Unaffected Area = 0.0 sq. m

Area of Cut = 217763.7 sq. m

Area of Fill = 260402.0 sq. m

Modified Area = 478165.7 sq. m

Max Z = 38.6 m

Min Z = 6.1 m

Z Range = 32.5 m

Cut Volume = 1,603,904 Cubic Metres

Fill Volume = 1,588,213 Cubic Metres

Fig.11 Surface-to-Surface Cut & Fill Volume Report for Client A

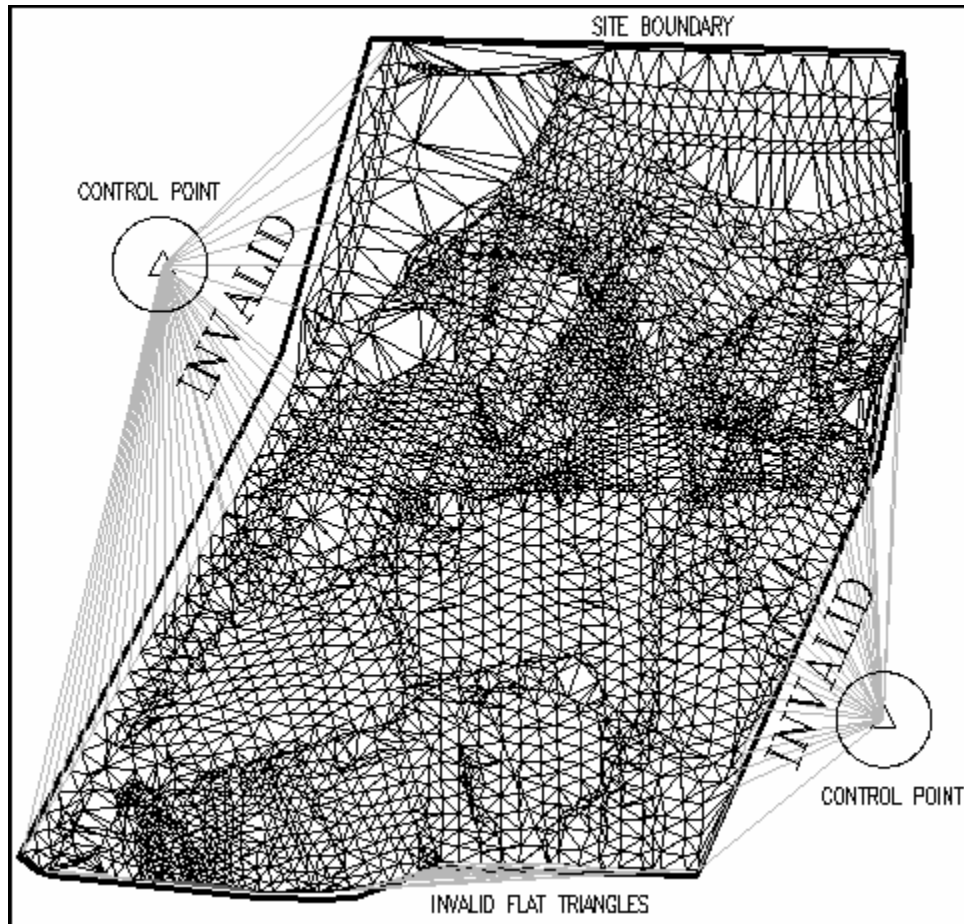


Fig.12 Blunders to Avoid in Digital Terrain Modeling: Invalid Triangles

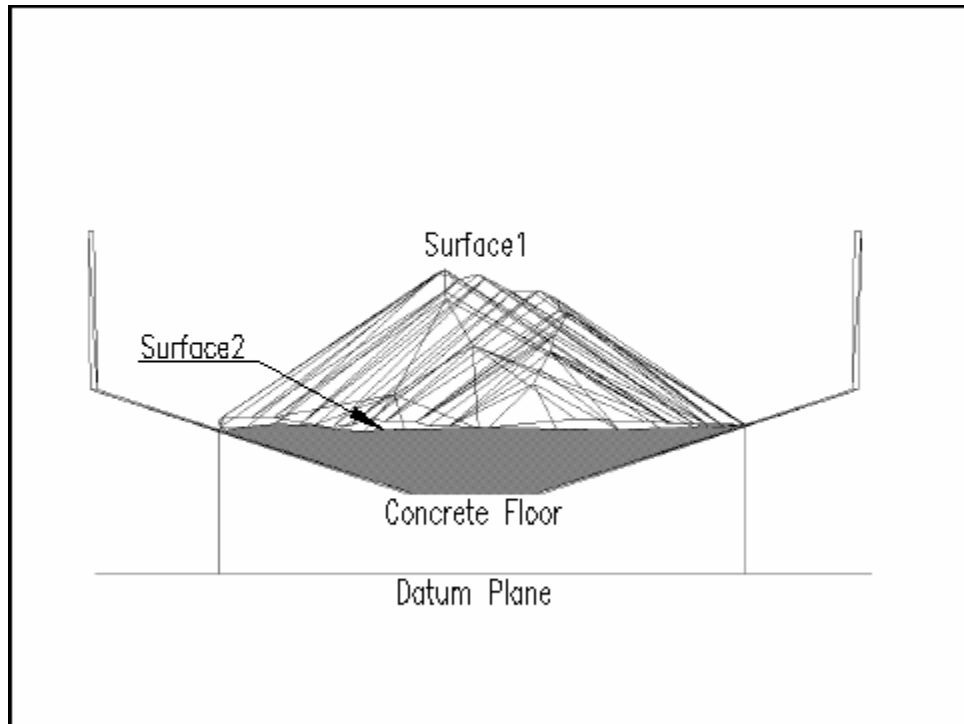


Fig.13 Failure of the Reference Datum Method for a Complex Stockpile

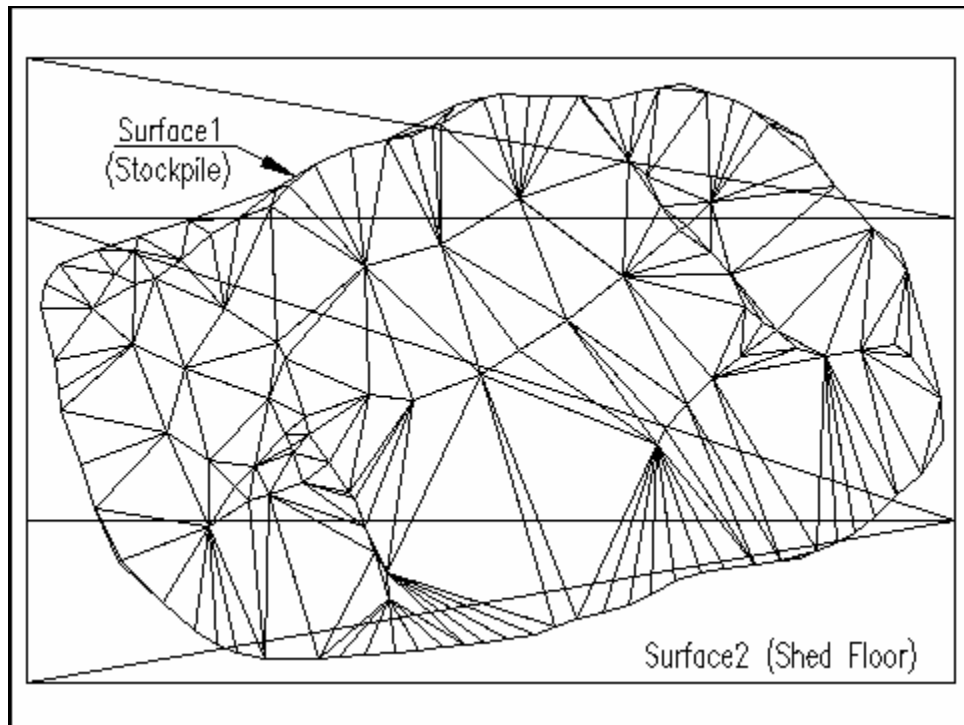


Fig.14 Two DTMs: The Stockpile (Delaunay) and the Shed Floor (Non-Delaunay)